

Stacked Central Configurations for Newtonian $N+4$ -Body Problems *

Furong Zhao^{1,2} and Shiqing Zhang¹

¹Department of Mathematics, Sichuan University, Chengdu,
610064, P.R.China

²Department of Mathematics and Computer Science, Mianyang Normal
University, Mianyang, Sichuan, 621000, P.R.China

Abstract: In this paper, we study spatial central configurations where N bodies are at the vertices of a regular N -gon T and the other 4 bodies are symmetrically located on the straight line that is perpendicular to the plane that contains T and passes through the center of T . We study the necessary conditions about masses for the bodies which can form a central configuration and show the existence of central configurations for Newtonian $N+4$ -body problems.

Keywords : $N+4$ -body problems, central configurations, stacked central configurations.

MSC: 34C15, 34C25.

1 Introduction and Main Results

The Newtonian n -body problems ([1], [23]) concern with the motions of n particles with masses $m_j \in R^+$ and positions $q_j \in R^3 (j = 1, 2, \dots, n)$, the

*This work is supported by NSF of China and Youth found of Mianyang Normal University.

motion is governed by Newton's second law and the Universal law:

$$m_j \ddot{q}_j = \frac{\partial U(q)}{\partial q_j}, \quad (1.1)$$

where $q = (q_1, q_2, \dots, q_n)$ and $U(q)$ is Newtonian potential:

$$U(q) = \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{|q_j - q_k|}, \quad (1.2)$$

Consider the space

$$X = \{q = (q_1, q_2, \dots, q_n) \in R^{3n} : \sum_{j=1}^n m_j q_j = 0\}, \quad (1.3)$$

i.e., suppose that the center of mass is fixed at the origin of the space. Because the potential is singular when two particles have same position, it is natural to assume that the configuration avoids the collision set $\Delta = \{q = (q_1, \dots, q_n) : q_j = q_k \text{ for some } k \neq j\}$. The set $X \setminus \Delta$ is called the configuration space.

Definition 1.1 ([20,24]): A configuration $q = (q_1, q_2, \dots, q_n) \in X \setminus \Delta$ is called a central configuration if there exists a constant λ such that

$$\sum_{j=1, j \neq k}^n \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, 1 \leq k \leq n. \quad (1.4)$$

The value of constant λ in (1.4) is uniquely determined by

$$\lambda = \frac{U}{I}, \quad (1.5)$$

Where

$$I = \sum_{k=1}^n m_k |q_k|^2. \quad (1.6)$$

Since the general solution of the n-body problem can't be given, great importance has been attached to search for particular solutions from the very

beginning. A homographic solution is a configuration which is preserved for all time. Central configurations and homographic solutions are linked by the Laplace theorem (see [24]). Collaps orbits and parabolic orbits have relations with the central configurations([17,19,20]). So finding central configurations becomes very important. The main general open problem for the central configurations is due to Winter[24] and Smale[22]: Is the number of central configurations finite for any choice of positive masses m_1, \dots, m_n ? Hampton and Moeckel([6]) have proved this conjecture for four any given positive masses.

For 5-body problem, Hampton ([5]) provided a new family of planar central configurations, called stacked central configurations which has some proper subset of three or more points forming a central configuration.

Ouyang, Xie and Zhang([15]) studied pyramidal central configurations for Newtonian $N+1$ -body problems; Zhang and Zhou([25]) studied double pyramidal central configurations for Newtonian $N+2$ -body problems; Mello and Fernandes([11]) studied new classes of spatial central configurations for the $N+3$ -body problem.

Based the above works, we study stacked central configuration for Newtonian $N+4$ -body problems. in $N+4$ -body problems, for which N bodies are at the vertices the vertices of a regular polygon, the other 4 bodies are symmetrically located on the straight line that is perpendicular to the plane that contains T and passes through the center of T , the vertical line passes the geometrical center of the regular polygon. (see Fig 1 for $N = 4$).

Related assumptions will be interpreted more precisely in the following.

Without loss of generality we can take a coordinate system such that

$$q_j = (\cos(\frac{(j-1)}{N}2\pi), \sin(\frac{(j-1)}{N}2\pi), 0) \text{ where } j = 1, \dots, N;$$

$$q_{N+1} = (0, 0, r_1), q_{N+2} = (0, 0, -r_1), q_{N+3} = (0, 0, r_2), q_{N+4} = (0, 0, -r_2).$$

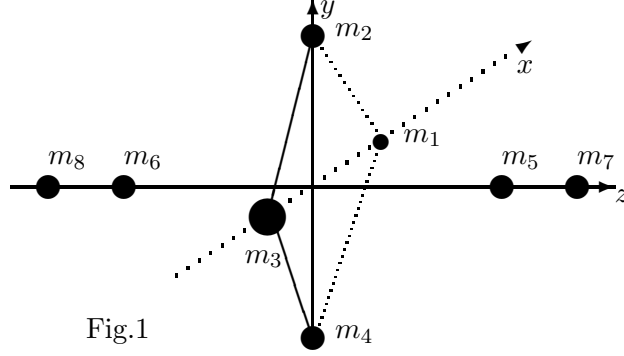


Fig.1

We have :

Theorem1.1: If $m_N + 1 = m_N + 2$ (or $m_N + 3 = m_N + 4$) and m_1, \dots, m_{N+4} form a central configuration, then

- (1): $\sum_{j=1}^N m_j q_j = 0$
- (2): $m_N + 3 = m_N + 4$ ($m_N + 1 = m_N + 2$).
- (3): m_1, \dots, m_N also form a central configuration.
- (4): $m_1 = \dots = m_N$.

Theorem1.2: Assume that $m_1 = \dots = m_N = 1, m_{N+1} = m_{N+2} = M_1, m_{N+3} = m_{N+4} = M_1$, then there exist $\epsilon(r_1, r_2) > 0$, $\delta > 0$ such that $\forall (r_1, r_2) \in \{(r_1, r_2) | r_2 > r_1 > \delta, r_2 - r_1 < \epsilon(r_1, r_2)\}$, we have positive masses $M_1 = M_1(r_1, r_2), M_2 = M_2(r_1, r_2)$ and all the $N + 4$ bodies form a central configuration.

Remark 1: $M_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}, M_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$.

Where :

$$\begin{aligned} a_{11} &= \frac{1}{4r_1^3} - \frac{2}{|1+r_1^2|^{3/2}}, a_{12} = \frac{1}{|r_1+r_2|^2 r_1} - \frac{1}{|r_1-r_2|^2 r_1} - \frac{2}{|1+r_2^2|^{3/2}}, \\ a_{22} &= \frac{1}{4r_2^3} - \frac{2}{|1+r_2^2|^{3/2}}, a_{21} = \frac{1}{|r_1+r_2|^2 r_2} + \frac{1}{|r_1-r_2|^2 r_2} - \frac{2}{|1+r_1^2|^{3/2}}, \\ b_1 &= \lambda^* - \frac{N}{|1+r_1^2|^{3/2}} \\ b_2 &= \lambda^* - \frac{N}{|1+r_2^2|^{3/2}} \end{aligned}$$

Remark 2: When $N = 2$, the **Theorem1.2** is related to the **Theorem 1.3** in [8].

2 The Proofs of Theorems

2.1 Some Lemmas

We need some Lemmas.

If $n \times n$ matrix $A = (a_{ij})$ satisfies

$$a_{i,j} = a_{i-1,j-1}, 1 \leq i, j \leq n, \quad (2.1)$$

where we assume $a_{i,0} = a_{i,n}, a_{0,j} = a_{n,j}$, then A is called a circulant matrix.

Lemma 2.1(see [10]). Let $A = (a_{ij})$ be a circulant matrix, then the eigenvalues λ_k and eigenvectors \vec{v}_k of A are

$$\lambda_k(A) = \sum_{j=1}^n a_{1,j} \rho_{k-1}^{j-1} \quad (2.2)$$

and

$$\vec{v}_k = (\rho_{k-1}, \rho_{k-1}^2, \dots, \rho_{k-1}^n)^T \quad (2.3)$$

where $\rho_k = e^{\sqrt{-1} \frac{2k\pi}{n}}$.

Lemma 2.2([24]): For $n \geq 3$, and $m_1 = m_2 = \dots = m_n$, if (m_1, m_2, \dots, m_n) locate at vertices of a regular polygon, then they form a central configuration.

From (1.4) and (1.3), notice that we have

$$\begin{aligned} \sum_{j=1, j \neq k}^n \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) &= -\lambda m_k q_k = -\lambda m_k (q_k - q_0) \\ &= -\lambda m_k (q_k - \frac{\sum_{j=1}^n m_j q_j}{M}) = -m_k \frac{\lambda}{M} \sum_{j=1}^n m_j (q_k - q_j) \end{aligned} \quad (2.4)$$

where $M = \sum_{j=1}^n m_j, q_0 = \frac{\sum_{j=1}^n m_j q_j}{M}$,

So (1.4) is also equivalent to

$$\sum_{j=1, j \neq k}^n m_j \left(\frac{1}{|q_j - q_k|^3} - \frac{\lambda}{M} \right) (q_j - q_k) = 0, k = 1, 2, 3, \dots, n. \quad (2.5)$$

2.2 The Proofs of Theorem 1.1 and Theorem 1.2

2.2.1 The Proof of Theorem 1.1

If m_1, \dots, m_{N+4} form a central configuration, we have

$$\sum_{j=1, j \neq k}^{N+4} m_j \left(\frac{1}{|q_j - q_k|^3} - \frac{\lambda}{M} \right) (q_j - q_k) = 0, k = 1, \dots, N+4. \quad (2.6)$$

Notice that (2.6) can be also written as :

$$\begin{aligned} & \sum_{j=1, j \neq k}^N m_j \left(\frac{1}{|q_j - q_k|^3} - \frac{\lambda}{M} \right) (q_j - q_k) + \\ & \sum_{j=1}^4 m_{N+j} \left(\frac{1}{|q_{N+j} - q_k|^3} - \frac{\lambda}{M} \right) (q_{N+j} - q_k) = 0, \\ & k = 1, \dots, N. \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} & \sum_{j=1}^N m_j \left(\frac{1}{|q_j - q_{N+l}|^3} - \frac{\lambda}{M} \right) (q_j - q_{N+l}) + \\ & \sum_{j=1, j \neq l}^4 m_{N+j} \left(\frac{1}{|q_{N+j} - q_{N+l}|^3} - \frac{\lambda}{M} \right) (q_{N+j} - q_{N+l}) = 0, \\ & l = 1, 2, 3, 4. \end{aligned} \quad (2.8)$$

Now (2.8) is taken inner product with vectors $\vec{e}_1 = (1, 0, 0)$ and $\vec{e}_2 = (0, 1, 0)$, then we get:

$$\begin{aligned} & \left(\frac{1}{|q_j - q_{N+l}|^3} - \frac{\lambda}{M} \right) \sum_{j=1}^N m_j \cos\left(\frac{(j-1)}{N} 2\pi\right) = 0 \\ & \left(\frac{1}{|q_j - q_{N+l}|^3} - \frac{\lambda}{M} \right) \sum_{j=1}^N m_j \sin\left(\frac{(j-1)}{N} 2\pi\right) = 0 \\ & j = 1, \dots, N. \end{aligned} \quad (2.9)$$

(2.9) can be also written as

$$\left(\frac{1}{|q_j - q_{N+l}|^3} - \frac{\lambda}{M}\right) \sum_{j=1}^N m_j q_j = 0, j = 1, \dots, N. \quad (2.10)$$

It is obvious that

$$\left(\frac{1}{|q_j - q_{N+l}|^3} - \frac{\lambda}{M}\right) = \left(\frac{1}{|q_k - q_{N+l}|^3} - \frac{\lambda}{M}\right), 1 \leq k, j \leq N, \quad (2.11)$$

we get

$$\sum_{j=1}^N m_j q_j = 0 \quad (2.12)$$

(2.8) is taken inner product with vector $\vec{e}_3 = (0, 0, 1)$, then we have:

$$\begin{aligned} & \sum_{j=1}^N m_j \left(\frac{1}{|1 + r_1^2|^{3/2}} - \frac{\lambda}{M} \right) r_1 + 0m_{N+1} + 2r_1 \left(\frac{1}{|2r_1|^3} - \frac{\lambda}{M} \right) m_{N+2} + \\ & (r_1 - r_2) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right) m_{N+3} + (r_1 + r_2) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right) m_{N+4} = 0 \end{aligned} \quad (2.13)$$

$$\begin{aligned} & \sum_{j=1}^N m_j \left(\frac{1}{|1 + r_1^2|^{3/2}} - \frac{\lambda}{M} \right) r_1 + 2r_1 \left(\frac{1}{|2r_1|^3} - \frac{\lambda}{M} \right) m_{N+1} + 0m_{N+2} + \\ & (r_1 + r_2) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right) m_{N+3} + (r_1 - r_2) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right) m_{N+4} = 0 \end{aligned} \quad (2.14)$$

$$\begin{aligned} & \sum_{j=1}^N m_j \left(\frac{1}{|1 + r_2^2|^{3/2}} - \frac{\lambda}{M} \right) r_2 + (r_2 - r_1) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right) m_{N+1} + \\ & (r_1 + r_2) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right) m_{N+2} + 0m_{N+3} + 2r_2 \left(\frac{1}{|2r_2|^3} - \frac{\lambda}{M} \right) m_{N+4} = 0 \end{aligned} \quad (2.15)$$

$$\begin{aligned} & \sum_{j=1}^N m_j \left(\frac{1}{|1+r_2^2|^{3/2}} - \frac{\lambda}{M} \right) r_2 + (r_1 + r_2) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right) m_{N+1} + \\ & (r_2 - r_1) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right) m_{N+2} + 2r_2 \left(\frac{1}{|2r_2|^3} - \frac{\lambda}{M} \right) m_{N+3} + 0m_{N+4} = 0 \end{aligned} \quad (2.16)$$

By(2.13)and(2.14),we have:

$$\begin{aligned} & 2r_1 \left(\frac{1}{|2r_1|^3} - \frac{\lambda}{M} \right) (m_{N+1} - m_{N+2}) + \\ & [(r_1 + r_2) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right) - (r_1 - r_2) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right)] (m_{N+3} - m_{N+4}) = 0 \end{aligned} \quad (2.17)$$

By(2.15)and(2.16),we have:

$$\begin{aligned} & [(r_2 - r_1) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right) - (r_1 + r_2) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right)] (m_{N+1} - m_{N+2}) + \\ & 2r_2 \left(\frac{1}{|2r_2|^3} - \frac{\lambda}{M} \right) (m_{N+4} - m_{N+3}) = 0 \end{aligned} \quad (2.18)$$

We define: $f(x) = x \left(\frac{1}{x^3} - \frac{\lambda}{M} \right)$, $\frac{df(x)}{dx} = -\frac{2}{x^3} - \frac{\lambda}{M} < 0$, so

$$f(r_2 - r_1) = (r_2 - r_1) \left(\frac{1}{|r_1 - r_2|^3} - \frac{\lambda}{M} \right) \neq (r_2 + r_1) \left(\frac{1}{|r_1 + r_2|^3} - \frac{\lambda}{M} \right) = f(r_2 + r_1) \quad (2.19)$$

If $m_{N+1} = m_{N+2}$, by (2.17) and (2.18), we have $m_{N+3} = m_{N+4}$.

If $m_{N+3} = m_{N+4}$, by (2.17) and (2.18), we have $m_{N+1} = m_{N+2}$.

By $m_{N+1} = m_{N+2}$, $m_{N+3} = m_{N+4}$ and (2.7), we have

$$\sum_{j=1, j \neq k}^N m_j \left(\frac{1}{|q_j - q_k|^3} - \frac{\lambda}{M} \right) (q_j - q_k) = 0, k = 1, \dots, N. \quad (2.20)$$

Since q_j locates on a unit circle, let

$q_k = \exp(\frac{2(k-1)\pi i}{N})$, $i = \sqrt{-1}$. By (2.20) we have

$$\sum_{j=1, j \neq k}^N m_j \left(\frac{1}{|q_{j-k} - 1|^3} - \frac{\lambda}{M} \right) (q_{j-k} - 1) = 0, k = 1, \dots, N. \quad (2.21)$$

We define the $N \times N$ matrix $C = (c_{k,j})$, where

$$c_{k,j} = 0, \text{ for } j = k; c_{k,j} = \left(\frac{1}{|q_{j-k}-1|^3} - \frac{\lambda}{M} \right) (q_{j-k} - 1), \text{ for } j \neq k.$$

C is circulant matrix since

$$c_{k-1,j-1} = c_{k,j} = 0, \text{ for } j = k; c_{k-1,j-1} = \left(\frac{1}{|q_{(j-1)-(k-1)}-1|^3} - \frac{\lambda}{M} \right) (q_{(j-1)-(k-1)} - 1) = \left(\frac{1}{|q_{j-k}-1|^3} - \frac{\lambda}{M} \right) (q_{j-k} - 1) = c_{k,j} \text{ for } j \neq k.$$

Then (2.21) can be written as

$$CM^* = 0 \quad (2.22)$$

where $M^* = (m_1, \dots, m_N)^T$.

By **Lemma2.1** and (2.22) we have

$$m_1 = m_2 = \dots = m_N. \quad (2.23)$$

By **Lemma2.2** and (2.23) we know that

m_1, \dots, m_N also form a central configuration.

The proof of **Theorem1.1** is completed.

2.2.2 The Proof of Theorem 1.2

Notice that (q_1, \dots, q_{N+4}) is a central configuration if and only if

$$\sum_{j=1, j \neq k}^{N+4} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, 1 \leq k \leq N+4. \quad (2.24)$$

Since the symmetries, (2.24) is equivalent to

$$\sum_{j=1, j \neq k}^{N+4} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, k = 1, N+1, N+3. \quad (2.25)$$

That is

$$\begin{aligned} -\lambda(1, 0, 0) = -\lambda^*(1, 0, 0) &+ \frac{(-1, 0, r_1)}{|1 + r_1^2|^{3/2}} M_1 + \frac{(-1, 0, -r_1)}{|1 + r_1^2|^{3/2}} M_1 \\ &+ \frac{(-1, 0, r_2)}{|1 + r_2^2|^{3/2}} M_2 + \frac{(-1, 0, -r_2)}{|1 + r_2^2|^{3/2}} M_2 \end{aligned} \quad (2.26)$$

$$-\lambda(0, 0, -r_1) = \frac{N(0, 0, r_1)}{|1 + r_1^2|^{3/2}} + \frac{(0, 0, 2r_1)}{|2r_1|^3}M_1 + \frac{(0, 0, r_1 + r_2)}{|r_1 + r_2|^3}M_2 + \frac{(0, 0, r_1 - r_2)}{|r_1 - r_2|^3}M_2 \quad (2.27)$$

$$-\lambda(0, 0, -r_2) = \frac{N(0, 0, r_2)}{|1 + r_2^2|^{3/2}} + \frac{(0, 0, r_1 + r_2)}{|r_1 + r_2|^3}M_1 + \frac{(0, 0, -r_1 + r_2)}{|r_1 - r_2|^3}M_1 + \frac{(0, 0, 2r_2)}{|2r_2|^3}M_2 \quad (2.28)$$

where λ^* such that

$$\sum_{j=1, j \neq k}^N \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda^* m_k q_k, k = 1, \dots, N.$$

(2.26),(2.27)and (2.28) are equivalent to

$$\lambda = \lambda^* + \frac{2}{|1 + r_1^2|^{3/2}}M_1 + \frac{2}{|1 + r_2^2|^{3/2}}M_2 \quad (2.29)$$

$$\lambda = \frac{N}{|1 + r_1^2|^{3/2}} + \frac{1}{4r_1^3}M_1 + \left(\frac{1}{|r_1 + r_2|^2 r_1} - \frac{1}{|r_1 - r_2|^2 r_1}\right)M_2 \quad (2.30)$$

$$\lambda = \frac{N}{|1 + r_2^2|^{3/2}} + \left(\frac{1}{|r_1 + r_2|^2 r_2} + \frac{1}{|r_1 - r_2|^2 r_2}\right)M_1 + \frac{1}{4r_2^3}M_2 \quad (2.31)$$

(2.29),(2.30)and (2.31) are equivalent to

$$\begin{aligned} & \left(\frac{1}{4r_1^3} - \frac{2}{|1 + r_1^2|^{3/2}}\right)M_1 + \left(\frac{1}{|r_1 + r_2|^2 r_1} - \frac{1}{|r_1 - r_2|^2 r_1} - \frac{2}{|1 + r_2^2|^{3/2}}\right)M_2 \\ & = \lambda^* - \frac{N}{|1 + r_1^2|^{3/2}} \end{aligned} \quad (2.32)$$

$$\begin{aligned} & \left(\frac{1}{|r_1 + r_2|^2 r_2} + \frac{1}{|r_1 - r_2|^2 r_2} - \frac{2}{|1 + r_1^2|^{3/2}}\right)M_1 + \left(\frac{1}{4r_2^3} - \frac{2}{|1 + r_2^2|^{3/2}}\right)M_2 \\ & = \lambda^* - \frac{N}{|1 + r_2^2|^{3/2}} \end{aligned} \quad (2.33)$$

when $a_{11}a_{22} - a_{12}a_{21} \neq 0$,we have

$$M_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}} \quad (2.34)$$

$$M_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \quad (2.35)$$

Where :

$$\begin{aligned} a_{11} &= \frac{1}{4r_1^3} - \frac{2}{|1+r_1^2|^{3/2}}, a_{12} = \frac{1}{|r_1+r_2|^2 r_1} - \frac{1}{|r_1-r_2|^2 r_1} - \frac{2}{|1+r_2^2|^{3/2}}, \\ a_{22} &= \frac{1}{4r_2^3} - \frac{2}{|1+r_2^2|^{3/2}}, a_{21} = \frac{1}{|r_1+r_2|^2 r_2} + \frac{1}{|r_1-r_2|^2 r_2} - \frac{2}{|1+r_1^2|^{3/2}}. \\ b_1 &= \lambda^* - \frac{N}{|1+r_1^2|^{3/2}} \\ b_2 &= \lambda^* - \frac{N}{|1+r_2^2|^{3/2}} \end{aligned}$$

If

$$a_{11} a_{22} - a_{12} a_{21} < 0, b_1 a_{22} - b_2 a_{12} < 0, b_2 a_{11} - b_1 a_{21} < 0, \quad (2.36)$$

then

$$M_1 > 0, M_2 > 0 \quad (2.37)$$

Notice that (2.36) is equivalent to

$$\frac{a_{11}}{a_{21}} < \frac{b_1}{b_2} < \frac{a_{12}}{a_{22}} \quad (2.38)$$

Notice that

$$\begin{aligned} \frac{a_{11}}{a_{21}} &= \frac{\frac{1}{4r_1^3} - \frac{2}{|1+r_1^2|^{3/2}}}{\frac{1}{|r_1+r_2|^2 r_2} + \frac{1}{|r_1-r_2|^2 r_2} - \frac{2}{|1+r_1^2|^{3/2}}} \\ &= \frac{|1+r_1^2|^{3/2} - 8r_1^3}{4r_1^3 \times |1+r_1^2|^{3/2}} \times \frac{(r_2^2 - r_1^2)^2 r_2 (1+r_1^2)^{3/2}}{2(r_1^2 + r_2^2)(1+r_1^2)^{3/2} - 2(r_2^2 - r_1^2)^2} \\ &= \frac{|1+r_1^2|^{3/2} - 8r_1^3}{4r_1^3} \times \frac{(r_2^2 - r_1^2)^2 r_2}{2(r_1^2 + r_2^2)(1+r_1^2)^{3/2} - 2(r_2^2 - r_1^2)^2} \end{aligned} \quad (2.39)$$

Since $\lim_{r_1 \rightarrow +\infty} \frac{|1+r_1^2|^{3/2} - 8r_1^3}{4r_1^3} = -\infty$, and for $r_2 = r_1, 2(r_2^2 - r_1^2)^2 r_2 = 0$. Then there exists $\delta_1 > 0, \epsilon(r_1, r_2) > 0$, such that for $r_1 > \delta_1$, we have

$$\frac{|1+r_1^2|^{3/2} - 8r_1^3}{4r_1^3} < 0,$$

and for $r_2 - r_1 < \epsilon(r_1, r_2)$, we have

$$2(r_1^2 + r_2^2)(1+r_1^2)^{3/2} - 2(r_2^2 - r_1^2)^2 r_2 > 0$$

So

$$\frac{a_{11}}{a_{21}} < 0, \forall (r_1, r_2) \in \{(r_1, r_2) | r_2 > r_1 > \delta, r_2 - r_1 < \epsilon(r_1, r_2)\} \quad (2.40)$$

We also notice that

$$\begin{aligned} \lim_{r_1 \rightarrow +\infty} \frac{b_1}{b_2} &= 1 \quad (2.41) \\ \frac{a_{12}}{a_{22}} &= \frac{4r_2(1+r_2^2)^{3/2} + 2(r_2^2 - r_1^2)^2}{(r_2^2 - r_1^2)^2(1+r_2^2)^{3/2}} \times \frac{4r_2^3(1+r_2^2)^{3/2}}{8r_2^3 - (1+r_2^3)^{3/2}} \\ &= \frac{16(1 + \frac{1}{r_2^2})^{3/2} + 8(1 - (\frac{r_1}{r_2})^2)}{(1 - (\frac{r_1}{r_2})^2)(8 - (1 + \frac{1}{r_2^2})^{3/2})} \end{aligned}$$

There exists $\delta_2 > 0$, such that for $r_2 > \delta_2 > 0$, we have

$$\frac{a_{12}}{a_{22}} > 2 \quad (2.42)$$

By (2.40), (2.41) and (2.42), there exists $\delta \geq \max\{\delta_1, \delta_2\}$, such that for $\delta < r_1 < r_2$ and $r_2 - r_1 < \epsilon(r_1, r_2)$, we have

$$\frac{a_{11}}{a_{21}} < \frac{b_1}{b_2} < \frac{a_{12}}{a_{22}}$$

The proof of **Theorem 1.2** is completed.

References

- [1] Abraham R. and Marsden J.E., Foundation of Mechanics, 2nd edn, Benjamin, New York, 1978.
- [2] Albouy A., The symmetric central configurations of four equal masses, Amer. Math. Soc, Providence, RI, 1996, 131-135.
- [3] Albouy A., Fu Y. and Sun S.Z., Symmetry of planar four-body convex central configurations, Proc. R. Soc. A 464(2008), 1355-1365.

- [4] Diacu F., The masses in a symmetric centered solution of the n -body problem, *Proc. AMS* 109(1990), 1079-1085.
- [5] Hampton M., Stacked central configurations: new examples in the planar five-body problem, *Nonlinearity* 18(2005), 2299-2304.
- [6] Hampton M., Moeckel R., Finiteness of relative equilibria of the four-body problem, *Invent. Math.*, 163(2)(2006) 289-312.
- [7] Lei J. and Santoprete M., Rosette central configurations, degenerate central configurations and bifurcations, *Celestial Mechanics and Dynamical Astronomy* 94(2006), 271-287.
- [8] Llibre J., Mello L.F., Triple and quadruple nested central configurations for the planar n -body problem, *Physica D* 238(2009) 563-571.
- [9] Long Y., Admissible shapes of 4-body non-collinear relative equilibria, *Adv. Nonlinear Stud.* 1(2003), 495-509.
- [10] Marcus M. and Minc H., A survey of matrix theory and matrix inequalities, Allyn and Bacon, Boston, 1964.
- [11] Mello L.F. and Fernandes A.C., New classes of spatial central configurations for the $n+3$ -body problem, *Nonlinear Analysis: Real World Applications* 12(2011) 723-730.
- [12] Moeckel R., On central configurations, *Math. Z.* 205(1990) 499-517.
- [13] Moeckel R., Simo C., Bifurcation of spatial central configurations from planar ones, *SIAM J. Math. Anal.* 26(1995): 978-998.
- [14] Moulton, F.R., The straight line solutions of the n -body problem, *Annals of Math.* 12(1910), 1-17.

- [15] Ouyang T.C.,Xie Z.F.and Zhang S.,Pyramidal central configurations and perverse solution,Electronic Journal of Differential Equations,106(2004),1-9.
- [16] Perko L.M.and Walter E.L.,Regular polygon solutions of N-body problem,Pro.AMS,94(1985),301-309.
- [17] Saari,D.G.,Singularities and collisions of Newtonian gravitational systems,Arch.Rational Mech.49(1973),311-320.
- [18] Saari,D.G.,On the role and properties of N body central configurations,Celestial Mechanics and Dynamical Astronomy 21(1980),9-20.
- [19] Saari,D.G.,On the manifolds of total collapse orbits and of completely parabolic orbits for the n-body problem,J.Diff.Eqs. 41(1981), 27-43.
- [20] Saari D.G.,Collisions,Rings and Other Newtonian N-body Problems,AMS Providence,Rhode Island.2005.
- [21] Shi J.,Xie Z.,Classification of four-body central configurations with three equal masses,J.Math.Anal.Appl.363(2010),512-524.
- [22] Smale S.,Mathematical problems for the next century,Math. Intelligencer 20(1998)141-145.
- [23] Smale S.,Topology and mechanics II,Inv.Math,11(1970),45-64.
- [24] WintnerA.,The analytical foundations of celestial mechanics, Princeton Univ. Press, 1941.
- [25] Zhang S.Q.and Zhou Q.,Double pyramidal central configurations,Physics Letters A,281(2001),240-248.